1.1 Quantum information, quantum computing, and HOQO Quantum information is a branch of science that approaches quantum mechanics from an information theory perspective. In this approach, key concepts in standard information theory such as bits, channels, circuits, and input/output relations are extended to fit the formalism of quantum theory. This information view of quantum mechanics provides important basis for quantum technologies and have already led to breakthroughs related to quantum foundations, quantum computation, and quantum communication. Among others, celebrated results of quantum information include the impossibility of cloning general quantum states [1], a protocol for teleporting quantum states [2], compressing communication via quantum superdensecoding [3], quantum-based cryptography [4], and device-independent protocols based on Bell nonlocality [5]. A quantum circuit is a model for quantum computation consisting of a sequence of quantum gates and quantum measurements. Decomposition of quantum circuits into elementary gates establishes the basis for analysing complexity in quantum computing and provide a useful framework to understand quantum information protocols. A typical example in terms of applications is the celebrated Shor's algorithm, a quantum-circuit that can be used to factorise integers efficiently [6]. Quantum operations form an important pillar of quantum theory and a key point for many applications to quantum technologies. Traditionally, quantum operations were only viewed as devices to transform quantum states, such as quantum communication channels between distant parties or quantum gate elements in a quantum circuit. However, quantum operations themselves may also be submitted to transformations and play the role of a state by a Higher-Order Quantum Operation (HOQO). A simple example of a higher-order is a quantum circuit with missing operations is illustrated in Fig. 1.

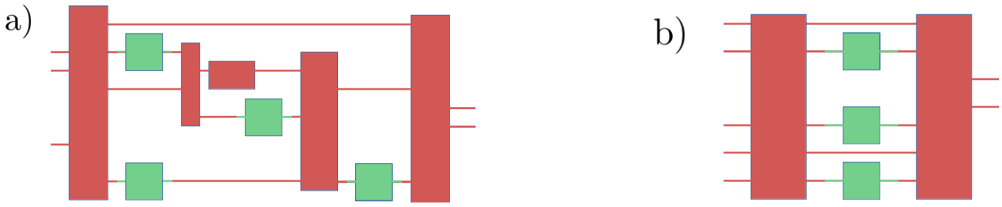

Figure 1: Pictorial representation of a quantum circuit. Elements in red represent a (casually ordered) HOQO which transforms input operations, in green, which may be plugged and unplugged in the circuit. Here a) is a general circuit and b) is parallel one where the input operations are used simultaneously.

The HOQO approach led the powerful mathematical methods to analyse quantum circuits, and problems involving quantum operations and quantum measurements. In particular, it allows various of such problems to be formalised as SemiDefinite Programs (SDP), and the symmetries which often appear may be analysed and treated with group representation theory methods. Causally ordered HOQO appeared in the literature under the name of quantum combs [7], quantum strategies [8], and quantum channels with memory [9], and led to important results in tasks such as quantum channel discrimination [10, 11], quantum metrology [12], tomography on quantum processes [13], controlling dynamics of quantum systems [14], universal transformation of unitary gates [15, 16], quantum causal-effect analysis [17] and optimal methods to store the action of quantum operations into a quantum memory state [18, 19]. Finally, we notice that, differently from states, operations have a clear notion of input and output, hence, when considering transformations between two or more operations, the concept of causal order emerges naturally. Interestingly, the postulates of quantum mechanics do not explicitly forbid the existence of HOQO which do not respect any definite causal order between the use of the input operations. This leaves room for the existence of quantum process with indefinite causality [20] and for a fruitful analysis on how causality should be understood in quantum theory.

1.2 Objective 1: Storing quantum operations in a quantum memory

The unitary storage-and-retrieve problem, also referred to as unitary learning, address the question of how to store a quantum operation U on a quantum state |ϕ⟩ so that the action of U might be retrieved in the future. In this context, the state 1 ⊗ U |ϕ⟩ where the action of U is stored is often referred to as a quantum memory state. The unitary operation U is arbitrary, and apart from its dimension, no extra assumption is made. For instance, one might consider the case where the operation U is a computer program capable of solving some particular problem, as in quantum oracle, or that U is a whole quantum circuit, or simply that U is some unitary dynamics which we would like to store it for later. It follows from the no-programming theorem [21] that quantum theory does not allow perfect unitary storage-and-retrieve. That is, if one stores an unknown operation on a quantum state, the operation obtained after the retrieval step is not going to be exactly the same one which was stored. However, one might look for optimal implementations, where the performance improves if we have access to multiple call of the operation we would like to store. The idea is that, we may apply k calls of U into part of a state |ϕ⟩, so that we might store the 1 ⊗ U^{⊗k} |ϕ⟩,and later, we perform a quantum operation R on 1 ⊗ U^{⊗k} |ϕ⟩ and an arbitrary input state |ψ⟩, so that we obtain a state which is U |ψ⟩, see Fig. 2. We may then have found better strategies by selecting the optimal storage state |ϕ⟩ and the optimal retrieval operation R.

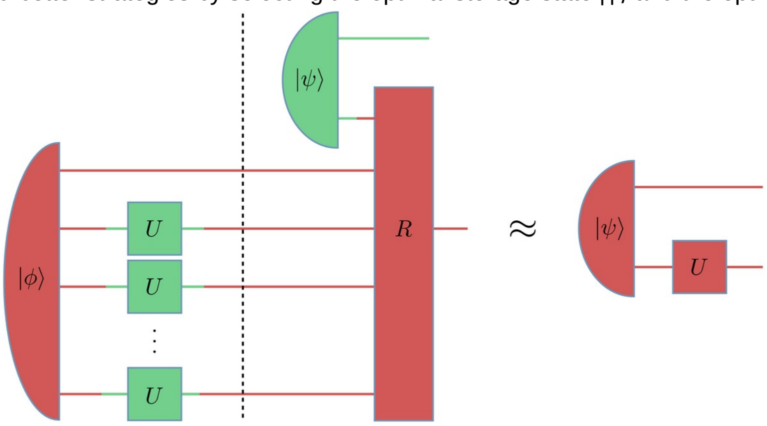

Figure 2: Pictorial representation of the storage-and-retrieve problem. A user prepares a state |ϕ⟩ to store k calls of some arbitrary unitary operation U . In a later moment, the user makes use of a quantum operation R to retrieve the usage of U on an arbitrary input state |ψ⟩.

The quantum storage-and-retrieval problem was analysed from different perspectives. For instance, Ref. [18] identifies the optimal state |ϕ⟩ identity the optimal storage state |ϕ⟩ and the optimal retrieval operation R for a deterministic non-exact case. And Ref. [19] which finds the optimal strategy for a probabilistic exact scenario. In this project, our goal is to consider a scenario where the input state |ψ⟩ where we desire to retrieve the action of U is known to the user. In all previous research, the quantum operation R used in the retrieval step is independent of the state |ψ⟩ where we intend to implement U . In this project, we consider a scenario where the state |ψ⟩ is known, for instance, if the operation U encodes some program, then, we might be the case where we intend to apply U on a particular known state |ψ⟩, this allows the retrieval operation R to be tailored for this particular input. Such extra-information might dramatically boost the performance of the known storage-and-retrieval protocols. Also, the decision to retrieve the operation U on a known or unknown state |ψ⟩ might be taken after U is stored in the memory state. Hence, one might keep the same storage step, and simply use a better retrieval protocol in the particular case where the target state was known. The main methods to solve and analyse this problem will be SDP and group representation theory, which are applicable via the HOQO approach. Our main goal is to obtain analytical results from SDP methods such as duality theory. However, for cases which are computationally feasible, we will also make a detailed numerical analysis, which should sharp our intuition and lead to specific results for simple scenarios e.g., qubits operations and small number of calls. All computational code used in this project will be uploaded to an online repository and will open and free for use.

1.3 Objective 2: HOQO for known input state

Similarly to the unitary storage-and-retrieve case, various HOQO might be analysed in a scenario where the final operation is performed in a known quantum states. For instance, Refs. [14–16, 22], study the problem of transforming k calls of an arbitrary unitary U into its inverse, transposition, and complex conjugation. We will then revisit these problems to see how much the knowledge of the input state might change the optimal performance. Similarly to before, we will consider deterministic non-exact and probabilistic exact protocols. Additionally, we will consider parallel strategies, where the all input operations U are performed in parallel, adaptive strategies, where we seek for sequential quantum circuits.

1.3 The big picture

This research project is part of a the bigger project of understanding and characterising HOQO, which we have been conducting for the last years in collaboration with various research over the world. One of the innovative aspects here is to consider scenarios where the states are known, and also to exploit the symmetries of strategies indefinite causality as a method to proved upper bounds for tasks involving quantum circuits. Since standard quantum circuits are causally ordered, the asymmetry of time and causal constrains make the problem considerably harder. These problems will require novel methods and innovative solutions, but some techniques such as group representation theory and semidefinite programming should work, at least for some particular instances of the problem.

1.5 The two supervisors

The French applicant, Marco Túlio Quintino, was a postdotoral researcher in the group of the Japanese supervisor, Prof. Mio Murao from 2016 to 2020. Since then Quintino and Murao have been regular collaborators working on the topic of higher-order quantum operations. This collaboration resulted to various articles published in high-impact journal with research conducted together with different students. For this reason, we believe that a joint supervision with a double degree should flow naturally. We also believe that having a co-cutela student should be very fruitful for all involved parties, the student, the supervisors, the groups, and the universities.

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